MODULATED RAYLEIGH-BÉNARD FERROCONVECTION IN COUPLE STRESS FLUID

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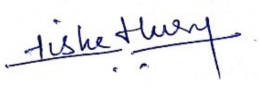
3. Dr. Nisha Mary Thomas (RAC Member 2) Signature : 

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Abstract

Ferromagnetic fluid with couple stress confined between two stress-free, isothermal horizontal plates separated by distance d is considered in this study. The effect of different types of modulation on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress is examined by considering sinusoidal (sine) and non-sinusoidal (square, triangular, and sawtooth) wave type of modulation. Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress subjected to gravity, rotation, temperature, and internal heat modulation is discussed by performing linear and non-linear analyses. The effect of various parameters on the onset of convection and heat transport is studied. The ferromagnetic parameters hasten the convection onset and enhance the heat transfer under various modulations. The results of a Newtonian fluid, ferromagnetic fluid, and couple stress fluid form the limiting cases of this study.

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1 Introduction

Fluid Mechanics mainly focuses on understanding, forecasting, and monitoring the fluid's behaviour. It comprises the relationship between physical properties, motions and forces in a continuous material. Fluid statics and fluid dynamics serve as two wings for fluid mechanics concentrating on the study of stationary fluids and flowing fluids, respectively. The study of fluid dynamics is vital due to its enormous applications in different engineering fields such as aeronautical, civil, marine, biomedical, and mechanical engineering. In addition, astrophysics, geophysics, biology, plasma physics and meteorology can also find the applications of fluid mechanics.

The fluids are categorised into Newtonian and non-Newtonian based on stress and the rate of strain relation. In Newtonian fluids, we commonly observe the linear association of stress with the rate of strain. Kerosine, gasoline and water are typical examples of Newtonian fluids. In non-Newtonian fluids, a non-linear association exists between stress and the rate of strain. The study of non-Newtonian fluids is quite challenging due to the complex relationship between stress and flow field. The standard non-Newtonian fluids are paint, blood, toothpaste, glues, chocolate, sand solution, shampoo, ink, nail polish etc.

The study in non-Newtonian fluids has received researcher's attention in the last century due to its application in modern technologies and industry. Micropolar fluids and couple stress fluids are the vital types of non-Newtonian fluids. The theory of Micropolar fluid and couple stress fluids was proposed by Eringen [1] and Stokes [2] respectively. Couple stress fluids is a unique case of micropolar fluids when microrotation balances with the natural vorticity of the fluid. The unique features of a couple stress fluids are the polar effect and capacity to possess large viscosity. Couple stress arises as a consequence of assuming that the mechanical action of one part of a body on another across a surface is equivalent to a force and moment distribution. Stokes theory is a generalization of the classical theory, which allows couple stresses, asymmetric stress tensor and body couples. The study of couple stress fluids is of great significance due to its scientific applications in lubrication and squeezing, polymer fluids extrusion, bio-fluid mechanics, etc.

Heat transfer in fluids predominately occurs through convection. Convective heat transfer plays a crucial role in various natural phenomenon, including solar and stellar structures. The first attempt to experimentally study the convective fluid flow was made by Bénard [3]. Rayleigh [4] independently developed an analytical method to examine the convection of onset. These studies resulted in the vast field of Rayleigh-Bénard convection. It is induced by the variation of density due to the temperature difference between the upper and lower plate. The researchers in the last century studied the Rayleigh-Bénard problem because of its analytical and experimental accessibility and for understanding complex flows in various industries.

In the last century, researchers paid less attention to realize the impact of magnetic and electric fields. However, in latest years, the interaction of electromagnetic fields with fluids enormously studied due to its escalating usefulness in medicine, nuclear fusion, chemical engineering, and high-speed noiseless printing. The fluid interaction

with various fields is divided into three main categories: Magnetohydrodynamics, Ferrohydrodynamics and Electrohydrodynamics.

- Magnetohydrodynamics is restricted to the study of the interaction of magnetic fields and electrically conducting fluids.
- Ferrohydrodynamics deals with the study of the impact of magnetic fields on non-conducting fluids.
- Electrohydrodynamics mainly focuses on the study of electric force effects on fluids.

Ferrohydrodynamics mainly focuses on the mechanics of the fluid motion affected by the vital forces of magnetic polarization. It involves magnetically active and electrically non-conducting fluids as a working medium. Ferrohydrodynamics, to some extent, varies from magnetohydrodynamics as the body force acting on fluid is polarization force instead of Lorenz force. The study of ferrohydrodynamics started acquiring attention after the synthesis of ferrofluids (commonly known as agnetic fluids/ferromagnetic fluids) by Papell [5].

1.1 Ferromagnetic fluids

Strongly magnetized fluids in the presence of an external magnetic field are known as ferromagnetic fluids. They are colloidal suspensions of surfactant coated monodomain magnetic particles dispersed in a non-conducting carrier liquid. Ferromagnetic fluids have to be artificially synthesized by considering 5% magnetic particles, 10% surfactant and 85% carrier liquid as they do not exist naturally.

- Magnetic particles: These particles are responsible for the magnetic properties in ferromagnetic fluids. Generally, iron, nickel and cobalt particles of nano-size are used as magnetic particles to synthesise ferromagnetic fluids. They behave as magnetic dipoles even in the absence of applied magnetic fields.
- Surfactant: It forms a protective layer around the magnetic particles and helps prevent agglomeration and maintain the colloidal consistency in suspension. The commonly used surfactants include oleic acid, sodium oleate, tetramethyl ammonium hydroxide, etc.
- Carrier liquid: It can either be polar or non-polar. It plays a significant role in determining the suitability of ferromagnetic fluid for different applications. Water, kerosene, blood, mineral oil, vacuum oil, silicon oil, etc., are typically used carrier liquids.

Rosensweig and Neuringer [6] first proposed the mathematical model for ferromagnetic fluids. Rosensweig [7] documented a detailed description of ferromagnetic fluids in his monograph. The incompressibility, homogeneity, body couple, polarization force and magnetic response are significant features of ferromagnetic fluids.

Magnetization in ferromagnetic fluids merely depends on density, magnetic field and temperature. Convection in a ferromagnetic fluid driven by a change in body force caused by variation of the magnetic field, temperature, and density is known as ferroconvection (also known as Rayleigh-Bénard convection in ferromagnetic fluids).

Ferroconvection has attracted researchers interset in the past five decades due to escalating usefulness in various industries.

1.2 Modulations

Monitoring convection is one of the foremost aims in many practical situations. Modulations serve as an effective tool to hasten or hinder the convection onset and heat transport. The modulation of gravity, rotation, temperature, and internal heat can be used to study Rayleigh-Bénard convection in ferromagnetic fluids.

1.2.1 Gravity modulation

The gravity modulation (or time-dependent body force) can be realized by vertically oscillating a fluid layer. It has been remarkably used to control the onset of convection and improve the heat transfer rate in space laboratory experiments.

1.2.2 Rotational modulation

Rotation modulation (or time-dependent Coriolis force) arises due to the uniform rotation of the fluid layer around the fixed z-axis in a time-periodic manner. It has applications in various engineering fields like chemical, industry, food industry, solidification of alloys and rotating turbomachinery.

1.2.3 Temperature modulation

Temperature modulation (or time-dependent boundary temperature) arises when the temperature profiles depend on space and time. It has widespread applications in the solidification of metallic alloys in crystal growth and material processing.

1.2.4 Internal heat modulation

Internal heat modulation arises when the arises when the internal heat profiles depend on space and time. It has auspisious applications in geophysical reactor safety analysis, nuclear fuel, metal waste, and storage of radioactive materials.

1.3 Waveforms

The different types of sinusoidal and non-sinusoidal waveforms considered to study the modulated Rayleigh-Bénard ferroconvection are sine, square, triangular and sawtooth. The sinusoidal sine waveform is a single frequency, whereas the non-sinusoidal waveforms like square, triangular and sawtooth are made up of multiple sinusoidal waveforms with different frequencies. The sinusoidal and non-sinusoidal waveforms are defined as follows:

- Sinusoidal wave form:

  Trigonometric sine wave form: $f(\omega, t) = f_1(\omega, t) = \sin(\omega t)$

• Non-Sinusoidal wave forms:

1. Square wave form: $f(\omega,t) = f_2(\omega,t) = \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{\sin(n\omega t)}{n}$,
2. Triangular wave form: $f(\omega,t) = f_3(\omega,t) = \frac{8}{\pi^2}\sum_{n=1,3,5,\cdots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin(n\omega t)$,
3. Sawtooth wave form: $f(\omega,t) = f_4(\omega,t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{[1-\cos(n\pi)]\cos(n\omega t)}{n^2}$.

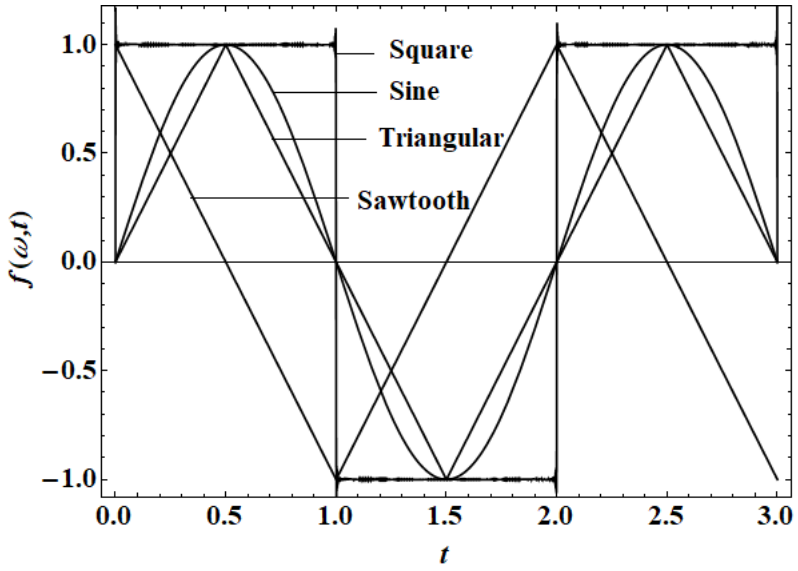


FIGURE 1: Schematic representation of wave forms for $\omega = 2\pi$

1.4 Need for the study

Convection in ferromagnetic fluids has been a fertile field of research due to its wide range of applications in regenerative magnetocaloric energy conversion, high-power capacity transformer system, vibration-dampers, magnetic fluid seals, shock absorbers, loudspeaker coils, chemical reactor, high-speed silent printers, energy conversion system, and biomedical applications (see [8–13]). The extensive applications of ferrofluids serve as background and motivation for this research work.

The promising applications of ferromagnetic fluids in various disciplines can be enhanced by considering ferromagnetic fluids with couple stress (see [14–17]). Ferromagnetic fluids with couple stress can be regarded as non-Newtonian in the sense that the stress tensor in the case of these liquids is asymmetric. The presence of asymmetric stress makes these problems more complex and challenging. Therefore, there is a need to study the convection and heat transport in a ferromagnetic fluid with couple stress.

The modulated Rayleigh-Bénard convection has gained a researcher's interest due to the broad spectrum of real-life situations. Modulations serve as an effective tool to postpone or prepone the convection onset and heat transport. The first experimental study by Donnelly [18] on modulated convection stimulated interest in this field. Many researchers in the near past have theoretically examined Rayleigh-Bénard convection in a ferromagnetic fluid with external constraints like gravity, rotation,

and temperature modulation. However, there are no reported works on modulated Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress. Hence, there is an urge to understand the impact of different modulations on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress.

The heat transport due to non-sinusoidal waveforms has importance in several electronic components and other industrial processes (see [19, 20]). From the literature, we observe Rayleigh-Bénard convection subjected to sinusoidal modulation is extensively studied, whereas non-sinusoidal modulation is sparsely studied. Considering non-sinusoidal modulations to examine the convection onset and heat transfer in ferromagnetic fluid with couple stress is mathematically and physically challenging. Therefore, there is a necessity to study Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress subjected to different sinusoidal and non-sinusoidal modulations.

1.5 Objective of the study

The main aim of this research work is to examine the impact of external constraints like gravity modulation, rotation modulation, temperature modulation, and internal heat modulation on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress. The individual impact of one sinusoidal (trigonometric sine) and three non-sinusoidal (square, triangular, and sawtooth) is considered to investigate the influence of different modulations on the onset of convection and heat transport in a ferromagnetic fluid with couple stress.

1.6 Methodology

The Rayleigh number and the correction Rayleigh number are premeditated by utilizing the regular perturbation method in the linear theory. The classical Venezian method is employed to obtain the eigenvalue. In the non-linear approach, the generalized Lorenz model is established using a truncated Fourier series for small-scale convection motion. The resulting non-autonomous Lorenz model is solved numerically to quantify the heat transport.

2 Review of literature

2.1 Rayleigh-Bénard convection in Newtonian ferromagnetic fluids

The ground-breaking work in the field of Rayleigh-Bénard convection in a ferromagnetic fluid was analyzed by Finlayson [21]. He showed that the spatial variation in magnetization causes convection in a ferromagnetic fluid. He employed the Galerkian technique to explain the concept of thermo-mechanical interaction in ferromagnetic fluids subjected to free-free and rigid-rigid. He found that the magnetic mechanism prevails over buoyancy mechanism only when the fluid layer is about 1 mm.

Gupta and Gupta [22] made a complete study on Rayleigh-Bénard convection in a rotating ferromagnetic fluid-saturated by a vertical magnetic field. They proved that for the Prandtl number greater than unity, overstability could not occur. Gotoh and Yamada [23] did linear stability analysis to investigate the Rayleigh-Bénard convection in a ferromagnetic fluid with the help of the Galerkin method. They discussed the impact of ferromagnetic parameters on the stability of the system.

Rudraiah and Sekhar [24] examined the role of internal heat on the onset of Rayleigh-Bénard convection in a ferromagnetic fluid. The higher-order Galerkin technique was employed to obtain the eigenvalue of the problem for free-free, rigid-free, and rigid-rigid isothermal boundary combinations. They denied the possibility of oscillatory convection by showing the validity of the principle of exchange of stability.

Venkatasubramanian and Kaloni [25] individually investigated the influence of Coriolis force on the onset of Rayleigh-Bénard convection in a ferromagnetic fluid subjected to stress-free boundaries, ferromagnetic boundaries, and rigid paramagnetic boundaries. They used the Galerkin method to obtain an expression for the critical Rayleigh number and found that Coriolis force stabilizes the system. They concluded that the type of the boundary determines the stabilization rate.

Auernhammer and Brand [26] performed linear and non-linear stability analyses to study the influence of rotation on Rayleigh-Bénard convection in a ferromagnetic fluid. They computed an expression for Rayleigh number for rigid and free boundaries and concluded that forward bifurcation exhibits in the system.

Aniss et al. [27] made an analytical study on the onset of Rayleigh-Bénard convection in a ferromagnetic fluid subjected to time-dependent Lorenz force. The linearized system is reduced to the Mathieu equation with damping term with the help of the first-order Galerkin method. They used the Floquet theory to obtain the eigenvalue of the problem for rigid-rigid and free-free cases. They discussed the criteria for the existence of harmonic and subharmonic modes at the onset.

Bajaj [28] analyzed the influence of gravity modulation on Rayleigh-Bénard convection in a ferromagnetic fluid. She obtained the parametric instability region by using the Floquet theory and discussed harmonic and subharmonic stability concepts. She concluded that gravity modulation could either stabilize or destabilize based on the different physical parameters.

Siddheshwar and Abraham [29] examined the effect of gravity modulation on Rayleigh-Bénard convection in a ferromagnetic fluid. Further, they analyzed the influence of synchronous and asynchronous temperature modulation on the onset of Rayleigh-Bénard convection in a ferromagnetic fluid. They used the Venezian method to perform linear stability analysis and found that Prandtl number and magnetization parameters display a reverse result at high frequencies.

Kaloni and Lou [30] used the Chebyshev pseudospectral method to study the thermal instability in a ferromagnetic fluid. Kaloni and Lou [31] analyzed the effect of Coriolis force on Rayleigh-Bénard convection in a ferromagnetic fluid by conducting linear and non-linear stability analysis. They employed a multiscale perturbation

technique to deduce the expression for the Nusselt number. They discussed the impact of rotation, magnetic field, and Prandtl number on heat transport.

Bajaj [32] was the first to determine the influence of two-frequency gravity modulation on Rayleigh-Bénard convection in a ferromagnetic fluid. They observed that harmonic and subharmonic instability simultaneously occurs at the bicritical point. Further, they concluded that the system's stability is affected by the variation in the concentration of magnetic particles.

Sunil and Mahajan [33] investigated the influence of rotation on heat transport in a horizontal layer of ferromagnetic fluid confined between two plates heated from below. They showcased the existence of subcritical instabilities by using the energy method. They discoursed the impact of Taylor number and ferromagnetic parameters on heat transport in the case of a stress-free boundary.

Laroze et al. [34] theoretically examined the Rayleigh-Bénard convection in a ferromagnetic fluid. They derived a generalized Lorenz model to discuss the bifurcation diagrams and Lyapunov exponents. They found that multiple transitions exhibit between chaotic and regular behaviors in the parameter space.

Mahajan et al. [35] examined the Rayleigh-Bénard convection in an internally heated ferromagnetic fluid. They performed linear stability analysis for different internal heat source models. Chebyshev Pseudospectral method was employed to obtain the solution of the eigenvalue problem.

Sibanda and Noreldin [36] considered the combined influence of Coriolis force and temperature modulation on Rayleigh-Bénard convection in a ferromagnetic fluid. They performed non-linear analysis to examine the impact of Coriolis force on heat transport in the presence of in-phase and out-of-phase temperature modulation. They have discussed the plots of isotherms and streamlines for different Rayleigh numbers.

Siddheshwar et al. [37] discussed the impact of rotation on Rayleigh-Bénard convection in a ferromagnetic fluid. They derived the Ginzburg Landau equation from the fifth-order Lorenz model by employing the method of multi-scales. They extracted the results on the impact of rotation on Rayleigh-Bénard convection in a dielectric fluid by launching a one-to-one correspondence among the governing equations of the two problems.

Melson et al. [38] presented the numerical study on Rayleigh-Bénard convection in a rotating ferromagnetic fluid. They derived a fourth-order magnetic Lorenz model with minimal Fourier series representation and obtained the Nusselt number expression. They concluded that the ferromagnetic parameters increase heat transport.

Aanam et al. [39] premeditated the impact of variable viscosity on Rayleigh-Bénard convection in a vertically oscillating ferromagnetic fluid. The tri-modal nonautonomous Lorenz model was computed to examine the thermorheological and magnetorheological influence on heat transport. They employed the Venezian method to obtain the expression for correction Rayleigh number from the linearized Lorenz model. They found that the ferromagnetic parameters enhance the heat transport and destabilize the system.

Aanam et al. [40] articulated the impact of temperature and magnetic field-dependent viscosity on Rayleigh-Bénard convection in a ferromagnetic fluid subjected to rotation modulation. They used Fourier series representation to obtain the fifth-order non-autonomous Lorenz system. The linearized Lorenz model was used to get the eigen value of the problem. They observed that modulated rotation and variable viscosity have reverse effects on the system's stability and heat transport.

2.2 Rayleigh-Bénard convection in couple stress fluids

Siddheshwar and Pranesh [41] made an analytical investigation on Rayleigh-Bénard convection in a couple stress fluid. They performed linear and non-linear analyses to discuss the effect of suspended particles on the onset and post-onset regimes of Rayleigh-Bénard convection. They addressed the validity of the principle of exchange of stability. The authors concluded that the couple stress parameter stabilizes the system.

Malashetty and Basavaraja [42] did a linear analysis to understand the influence of gravity/temperature modulation on the onset of Rayleigh-Bénard convection in a couple stress fluid. The correction Rayleigh number in both cases was dependent on modulation frequency. They concluded the stability of the system could be controlled by the proper choice of frequency and amplitude of modulation.

Pranesh and George [43] investigated the combined influence of temperature modulation and magnetic field on Rayleigh-Bénard convection in a couple stress fluid. The Venezian method was used to perform linear stability analysis. The impact of in-phase, lower wall, and out-of-phase temperature modulation on the convection onset was discussed. They concluded that out-of-phase modulation enhances the system's stability.

Pranesh and George [44] examined the effect of both electric field and temperature modulation on Rayleigh-Bénard convection in a couple stress fluid. The eigenvalue was obtained by employing the Venezian method for three oscillating temperatures: in phase, lower wall, and out-of-phase temperature modulation. A detailed discussion on the effect of various parameters on the stability of the system is presented. It is seen that subcritical motion arises due to in-phase modulation, and supercritical motion arises in out-of-phase temperature modulation and gravity modulation.

Pranesh and George [45] was the first to examine the impact of rotation modulation on Rayleigh-Bénard convection in a couple stress fluid. The Venezian method was employed to calculate the shift in critical Rayleigh number due to rotation modulation. The modulation frequency, Taylor number, couple Stress parameter, and Prandtl number were found to affect the system's stability. It is seen that observed that rotational modulation leads to hinders the onset of convection.

Thomas and George [46] made a comprehensive study to understand the impact of internal heat generation on Rayleigh-Bénard convection in a vertically oscillating couple stress fluid. The classical Fourier heat flux was replaced by Maxwell–Cattaneo law in this study. The correction Rayleigh number was computed by the regular perturbation method to examine the effect of various parameters on the stability of

the system. They found that gravity modulation stabilizes the system and internal heat destabilizes the system.

Keshri et al. [47] analyzed non-linear magneto-convection in a couple stress fluid subjected to magnetic field modulation with internal heat. The cubic Landau equation is derived in terms of amplitude and numerically solved to determine heat transport. It is observed that the couple stress parameter and internal Rayleigh number enhance the heat transport.

Aanam and Pranesh [48] investigated the role of synchronous and asynchronous temperature modulation on Rayleigh-Bénard convection in a couple stress fluids. The Ginzburg-Landau equation was derived to study the amount of heat transport. They discussed the streamlines plots for synchronous and asynchronous temperature modulation and concluded that the Chandrasekhar number diminishes the heat transport.

2.3 Modulated convection in Newtonian/non-Newtonian fluids

Venezian [49] was the first to investiage the impact of modulation on thermal convection in a Newtonian fluid. He premeditated an expression for the correction Rayleigh number as a function of modulation frequency. He reported that the onset of convection could be restrained by fine-tuning of amplitude and frequency of modulation.

Gresho and Sani [50] did innovative work in the field of thermal instability in a vertically oscillating fluid heated from below. They performed linear stability analysis to examine the role of sinusoidal gravity modulation on stability. They discoursed the impact of finite-amplitude flows.

Rosenblat and Herbert [51] carried out the thermal instability subjected to low-frequency time-periodic boundary temperature. They obtained an asymptotic solution and discussed the behavior of infinitesimal disturbances. Rosenblat and Tanaka [52] used the Floquet theory to examine the effect of thermal modulation on the onset of RBC. They concluded that the Rayleigh number enhancement occurs due to temperature modulation.

Bhattacharjee [53] was the first to analyze the effect of rotation speed modulation on Rayleigh-Bénard convection. Galerkin truncation is used to examine the influence of rotation speed modulation on the convection onset. They concluded that the convective threshold could be controlled by altering rotation speed and Prandtl number.

Siddheshwar and Pranesh [54] analyzed the influence of time-periodic body force/ temperature on the onset of Rayleigh-Bénard convection in a micropolar fluid. The correction Rayleigh number was derived by using the Venezian method. They concluded that gravity modulation would post-pone the convection onset, and temperature modulation results in sub-critical motion.

Siddheshwar [55] reported the first work on the gravity modulation problem using the Ginzburg-Landau model. Bhadauria et al. [56] premeditated the impact of time-periodic gravity/temperature on Rayleigh-Bénard convection in a rotating Newtonian fluid. They performed weakly non-linear analysis to examine the effect of gravity/temperature on heat transport. The stationary mode of convection was considered to arrive at the Ginzburg -Landau equation. They found that heat transport is diminished/enhanced by gravity /temperature/ modulation.

Siddheshwar et al. [57] made a brief study on stationary magneto-convection in a Newtonian liquid subjected to imposed time-periodic body force/boundary temperature. The effect of modulations on heat transport was studied by deriving Ginzburg–Landau equation. Quantification of heat transport was done by solving the Ginzburg–Landau equation. The authors discussed the role of various parameters on heat transport.

Siddheshwar and Kanchana [58] discussed the impact of trigonometric sine, square and triangular wave-types of gravity modulation on Rayleigh–Bénard convection in a Newtonian liquid and Newtonian nanoliquids. The authors have employed Buongiorno two-phase model to examine convection in nanoliquids. The multiscale method was used to obtain Ginzburg–Landau equation from the five-mode Lorenz model. The authors have explained the role of nanoparticles in heat transport enhancement. Further, they found that trigonometric sine, square and triangular wave-types of gravity modulation hinder convection onset and diminish heat transport.

Siddheshwar and Meenakshi [59] examined the influence of time-periodic body force on Rayleigh-Bénard convection in nanoliquids. Further, they have analyzed the role of sinusoidal and non-sinusoidal gravity modulation to study the onset and post-onset regimes of Rayleigh–Bénard convection. They anticipated the amount of heat transfer for twenty nanoliquids and concluded that square type of gravity modulation enhances and triangular type of gravity modulation diminishes the heat transport.

Mathew et al. [60] examined the combined effect of the concentration modulation and magnetic field on solutal Rayleigh-Bénard convection in a micropolar fluid using linear and non-linear analysis. They arrived at the expression of the correction Rayleigh number by using the regular perturbation method. They used the multi-scale approach to reduce the four-mode Lorenz model to Ginzburg-Landau equations and discussed the impact of concentration modulation on mass transport.

Kanchana et al. [61] contemplated the individual impact of gravity, temperature, and rotation modulation on Rayleigh-Bénard convection in Newtonian liquids and nanoliquids. They performed non-linear stability analysis to examine the amount of heat transport in twenty-eight nanoliquids using truncated Fourier series representation. The authors employed the multi-scale method to arrive at Ginzburg–Landau equations from the generalized Lorenz model. They found that rotation and gravity modulation are affected most by modulation frequency, whereas temperature modulation is affected most by phase angle and frequency of modulation.

Gupta et al. [62] considered the impact of magnetic field and rotation modulation on Rayleigh Bénard convection in a fluid. They calculated the amount of heat transfer by solving the Ginzburg Landau equation for its amplitude. They showed that the Taylor number diminishes the heat transport.

Pranesh et al. [63] examined the g-jitter effect on a three-component system in a Newtonian liquid. They made linear and non-linear analyses to investigate the impact of sinusoidal or non-sinusoidal non-uniform vertical vibrations on the onset of convection and heat and mass transport. They concluded that all four types of time-periodic body force hinders the convection onset and diminishes heat transport.

Kumar et al. [64] did the non-linear analysis to investigate the influence of time-periodic body force on solutal Rayleigh-Bénard convection in an electrically conducting Newtonian liquid. The effect of gravity modulation on mass transport is studied using the cubic Landau equation. They concluded that gravity modulation enhances mass transport.

Jayalatha and Muchikel [65] studied the role of gravity modulation on Rayleigh-Bénard convection in viscoelastic ferromagnetic fluids. They discussed the amount of heat transport by solving the four-mode Khayat–Lorenz model for its amplitude. They obtained the results of Newtonian, Rivlin–Ericksen fluids, and Maxwell fluids as limiting study cases.

Meghana and Pranesh [66] discussed the impact of concentration modulation on two-component convection in a micropolar fluid. They performed the linear and non-linear stability analysis to determine the onset of convection and heat and mass transport. They deduced the Ginzburg Landau equation from the six mode non-autonomous Lorenz model and discoursed the influence of non-dimensional parameters on heat and mass transport. They found that the onset of convection is controlled by fine-tuning frequency and amplitude of modulation.

3 Individual effects of sinusoidal and non-sinusoidal gravity modulation on Rayleigh-Bénard convection in a ferromagnetic liquid with couple stress

This section examines the impact of the time-dependent body force (see section 1.2.1) in a ferromagnetic liquid with couple stress using linear and non-linear analyses. One sinusoidal (sine) and three non-sinusoidal (square, triangular, and saw-tooth) wave types of gravity modulation are considered to investigate the effect of gravity modulation on the convection onset and heat transport. Vertically oscillating a ferromagnetic liquid with couple-stress achieves gravity modulation in the system and is represented by $\vec{g}(t) = -g_o[1 + \epsilon f(\gamma, t)]\hat{k}$.

3.1 Mathematical formulation

The Newtonian-Boussinesq ferromagnetic liquid with couple stress is considered between two parallel plates which are at a distance d from each other. A uniform

magnetic field $\vec{H}$ is applied in the vertical direction and time-dependent gravity is acting vertically downwards. The temperature gradient is maintained by heating from below (see Figure 2).

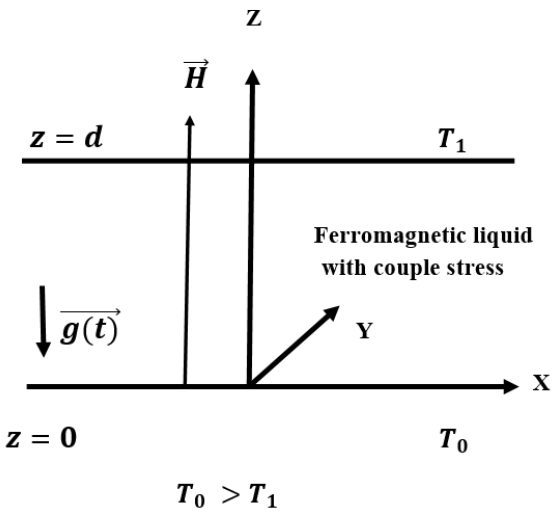


FIGURE 2: Physical configuration

The conservation equations of mass, linear momentum and energy, the equation of state and Maxwell's equation in the presence of the applied magnetic field, constitutive equation and magnetic equation of state are (see [28, 41])

$$\nabla \cdot \vec{q} = 0, \tag{3.1}$$

$$\left.\begin{aligned} \rho_o \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla p - \rho g_o[1+\epsilon f(\gamma,t)]\hat{k} + \mu_o(\vec{M}.\nabla)\vec{H} \\ +\mu\nabla^2\vec{q} - \mu'\nabla^4\vec{q} \end{aligned}\right\}, \tag{3.2}$$

$$\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T = \chi\nabla^2 T, \tag{3.3}$$

$$\rho = \rho_o[1-\alpha_t(T-T_o)], \tag{3.4}$$

$$\nabla\cdot\vec{B} = 0, \qquad \nabla\times\vec{H} = \vec{0}, \tag{3.5}$$

$$\vec{B} = \mu_o(\vec{H}+\vec{M}), \tag{3.6}$$

$$\vec{M} = \frac{\vec{H}}{H}\left[M_o + \chi_m(H-H_o) - \chi_T(T-T_o)\right], \tag{3.7}$$

where $\vec{q}$ is the velocity, ρ is the density of the fluid, p is the pressure, T is the temperature, α_t is the coefficient of thermal expansion, μ_o is the magnetic permeability, $f(\gamma,t)$ represents the sinusoidal and non-sinusoidal wave forms of gravity modulation, ϵ is the amplitude of gravity modulation, γ is the frequency of gravity modulation, $\hat{k}$ is the unit vector along the z-axis, t is the time and g_o is the acceleration due to gravity in the absence of modulation, χ is the thermal diffusivity, $\vec{H}$ is the magnetic field intensity, $\vec{M}$ is the magnetization, $\vec{B}$ is the magnetic induction, μ is the dynamic viscosity, μ' is the couple-stress viscosity, $\chi_T = -\left(\partial\vec{M}/\partial T\right)_{(H_o,\,T_o)}$

is the pyromagnetic coefficient and $\chi_m = \left(\partial \vec{M}/\partial H\right)_{(H_o,\ T_o)}$ is the magnetic susceptibility.

The dimensionless governing equations are given by:

$$\left.\begin{aligned}\frac{1}{Pr}\left[\frac{\partial}{\partial t}(\nabla^2\psi)\right] = -R\left[1+\epsilon\ f(\omega,t)\right]\frac{\partial T}{\partial x} + RM_1\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial T}{\partial x}\right] + \nabla^4\psi \\ -C\nabla^6\psi + RM_1\ J\left(T, \frac{\partial\phi}{\partial z}\right) + \frac{1}{Pr}J(\psi, \nabla^2\psi)\end{aligned}\right\}, \tag{3.8}$$

$$\frac{\partial T}{\partial t} = \nabla^2 T - \frac{\partial\psi}{\partial x} + J(\psi, T), \tag{3.9}$$

$$M_3\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial T}{\partial z} = 0, \tag{3.10}$$

where $Pr = \dfrac{\mu}{\rho_o\chi}$ is the Prandtl number, $R = \dfrac{\rho_o\alpha_t g_o\Delta T d^3}{\mu\chi}$ is the Rayleigh number, $C = \dfrac{\mu'}{\mu d^2}$ is the Couple stress parameter, $M_1 = \dfrac{\mu_o\chi_T^2\Delta T}{d\alpha_t\rho_o g_o(1+\chi_m)}$ is the buoyancy magnetization parameter, $M_3 = \dfrac{1+\frac{M_o}{H_o}}{1+\chi_m}$ is the non-buoyancy magnetization parameter and $\omega = \dfrac{\gamma d^2}{\chi}$ is the dimensionless frequency of modulation.

The four different types of gravity modulation are given by:

1. Gravity modulation by trignometric sine waveform: $f(\omega,t) = \sin(\omega t)$,
2. Gravity modulation by square waveform: $f(\omega,t) = \sum_{n=1,3,5,\cdots}^{\infty}\dfrac{4\ \sin(n\omega t)}{n\pi}$,
3. Gravity modulation by triangular waveform: $f(\omega,t) = \sum_{n=1,3,5,\cdots}^{\infty}\dfrac{8\ (-1)^{\frac{n-1}{2}}}{n^2\pi^2}\sin(n\omega t)$,
4. Gravity modulation by sawtooth waveform: $f(\omega,t) = \sum_{n=1}^{\infty}\dfrac{4\ [1-\cos(n\pi)]\cos(n\omega t)}{n^2\pi^2}$.

The above defined sinusoidal and non-sinusoidal waves forms are graphically represented in the figure 1.

Equations (3.8)–(3.10) are solved subjected to the stress-free, isothermal, and magnetic boundary conditions based on the assumption of infinite magnetic susceptibility boundary conditions:

$$\psi = \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^4\psi}{\partial z^4} = T\ = \frac{\partial\phi}{\partial z} = 0 \qquad at \qquad z = 0, 1. \tag{3.11}$$

3.2 Linear stability analysis

Linearized Lorenz model for studying the effect of gravity modulation on the onset of convection is given by:

$$\frac{dA_1}{dt} = -\frac{R\pi\alpha Pr}{k^2}\left[\frac{(1+M_3\alpha^2)[1+\epsilon f(\omega,t)]+M_1M_3\alpha^2}{1+M_3\alpha^2}\right]A_2 - Pr\eta k^2 A_1, \tag{3.12}$$

$$\frac{dA_2}{dt} = -k^2\ A_2 - \pi\alpha\ A_1. \tag{3.13}$$

where $\eta = 1 + k^2\ C, \quad k^2 = \pi^2(\alpha^2+1)$.

Following the multiscale method (see Siddheshawar and Kanchana [58]), we get

$$R_o = \frac{k^6\eta}{\pi^2\alpha^2\mathcal{M}}, \tag{3.14}$$

$$R_1 = 0, \tag{3.15}$$

$$R_2 = \frac{R_o^2\pi^2\alpha^2 Pr\epsilon^2 Y_1\mathcal{I}}{k^2\mathcal{M}[Y_1^2+Y_2^2]}, \tag{3.16}$$

where,

$$Y_1 = \omega^2 - k^4\eta Pr + \frac{R_o\pi^2\alpha^2 Pr\mathcal{M}}{k^2}, \quad Y_2 = k^2\omega(1+\eta Pr),$$

$$\mathcal{M} = \frac{1+M_3\alpha^2+M_1M_3\alpha^2}{1+M_3\alpha^2}, \qquad \mathcal{I} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} f^2(\omega,t)\ dt.$$

TABLE 1: Limiting cases in the absence of gravity modulation

Parameter	Critical Rayleigh number	Fluid	Author
$\mathcal{M} = \eta = 1$	$R_{oc} = \frac{k^6}{\pi^2\alpha^2}$	Newtonian fluid	Rayleigh [4]
$\eta = 1; \mathcal{M} \neq 1$	$R_{oc} = \frac{k^6}{\pi^2\alpha^2\mathcal{M}}$	Newtonian Ferromagnetic fluid	Finlayson [21]
$\mathcal{M} = 1; \eta \neq 1$	$R_{oc} = \frac{k^6\eta}{\pi^2\alpha^2}$	Couple stress fluid	Siddheshwar and Pranesh [41]

The critical Rayleigh number, R_c , is considerd to examine the effect of non-dimemsional parameters on the onset of convection and is given by:

$$R_c = R_{oc} + \epsilon^2 R_{2c}, \tag{3.17}$$

where R_{oc} is the unmodulated critical Rayleigh number and R_{2c} is the correction Rayleigh number.

3.3 Non-linear stability analysis

The effect of gravity modulation on the heat transport is investigated by solving the Lorenz model using the double Fourier series representation of χ, T and ϕ:

$$\frac{dA}{dt} = -Pr\eta A + Pr[1 + \epsilon\ f(\omega, t)]B + Pr\lambda\ B + \Lambda BD, \tag{3.18}$$

$$\frac{dB}{dt} = -B + R'A - AC, \tag{3.19}$$

$$\frac{dD}{dt} = -bD + AB, \tag{3.20}$$

where $b = \dfrac{4\pi^2}{k^2}$, $R' = \dfrac{\pi^2\alpha^2 R}{k^6}$, $\lambda = \dfrac{M_1 M_3 \alpha^2}{1 + M_3\alpha^2}$, and $\Lambda = \dfrac{M_1 Pr}{(1 + M_3\alpha^2)R'}$.

The classical Lorenz model [67] for thermal convection in a Newtonian fluid is obtained by setting $M_1 = C = \epsilon = 0$ in equations (3.18)-(3.20).

Average Nusselt number is considered in order to study the effect of various parameters on heat transfer and is given by:

$$\overline{Nu} = \left(\frac{\omega}{2\pi}\right) \int_0^{\frac{2\pi}{\omega}} Nu(t)\ dt, \tag{3.21}$$

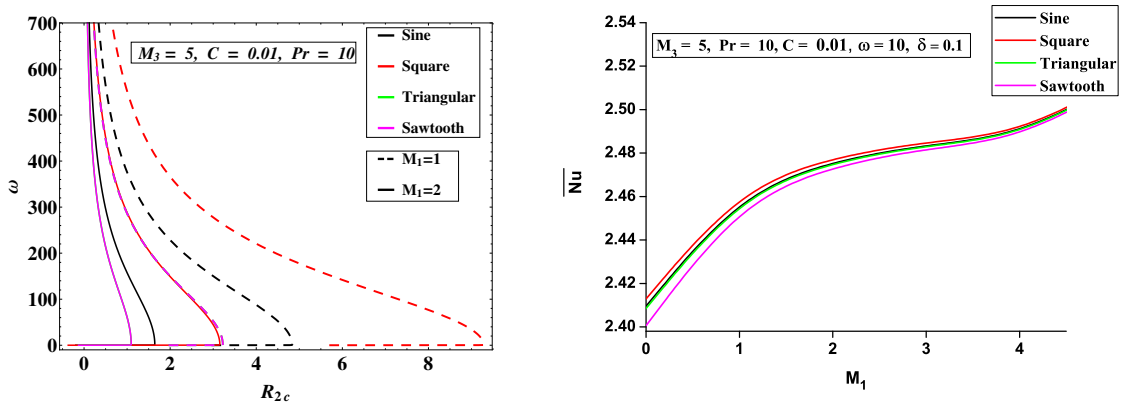
where

$$Nu(t) = 1 + \frac{2}{R'} D(t).$$

3.4 Results

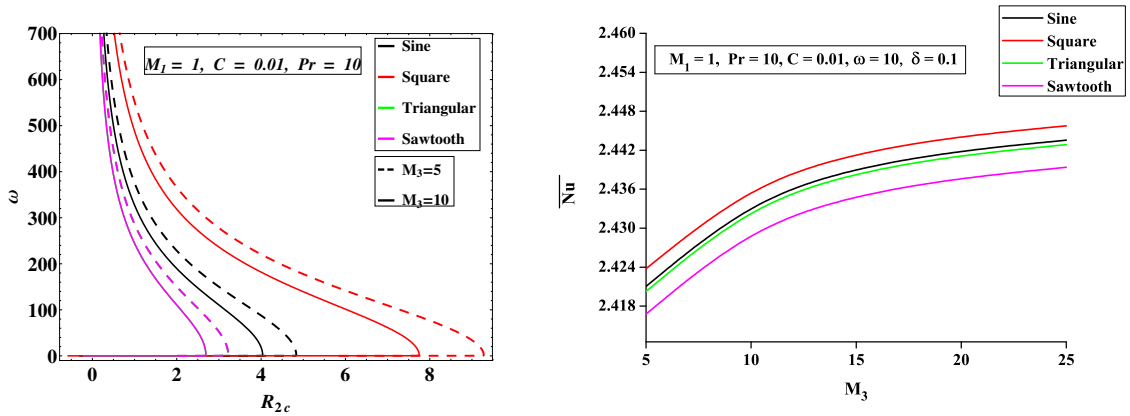
The following conclusions are drawn from the study:

1. Gravity modulation in a ferromagnetic liquid with couple stress stabilizes the system and diminishes heat transport in comparision with the unmodulated system.
2. It is possible to control the onset of convection and heat transport by choosing any of the four wave forms of gravity modulation.
3. Increase in the value of the couple stress parameter C stabilizes the system and decreases the heat transport.
4. Increase in M_1 and M_3 destabilizes the system.
5. $\overline{Nu}$ increases with increase in M_1, M_3 and Pr.
6. Square wave form is more stable compared to sine, triangular and sawtooth wave forms, i.e., $R_c^{\text{Square}} > R_c^{\text{Sine}} > R_c^{\text{Triangular}} = R_c^{\text{Sawtooth}}$.
7. Square wave form enhances the heat transport compared to sine, triangular and sawtooth wave forms, i.e., $\overline{Nu}^{\text{Square}} > \overline{Nu}^{\text{Sine}} > \overline{Nu}^{\text{Triangular}} > \overline{Nu}^{\text{Sawtooth}}$.
8. Results on gravity modulation in Newtonian fluids, ferromagnetic fluids and couple stress fluids can be obtained as a limiting case of the present problem.



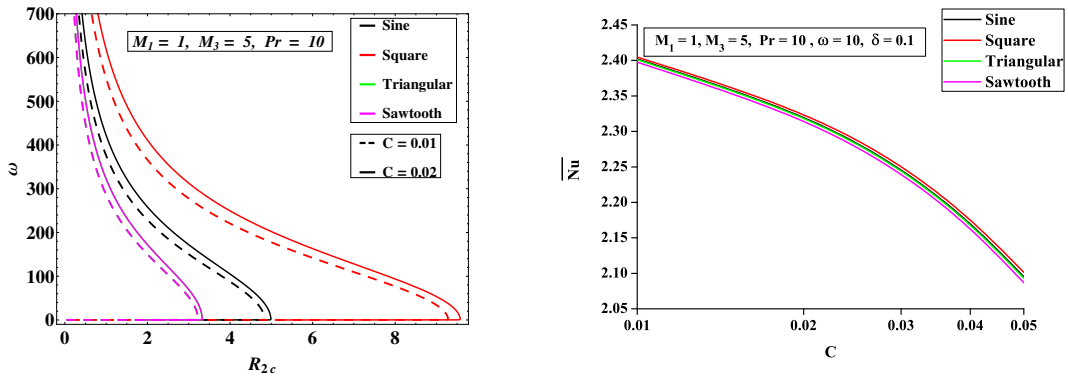
(A) (B)

FIGURE 3: Plot of (A) R_{2c} versus ω for different values of M_1 (B) $\overline{Nu}$ versus M_1 for sinusoidal and non-sinusoidal gravity modulation.



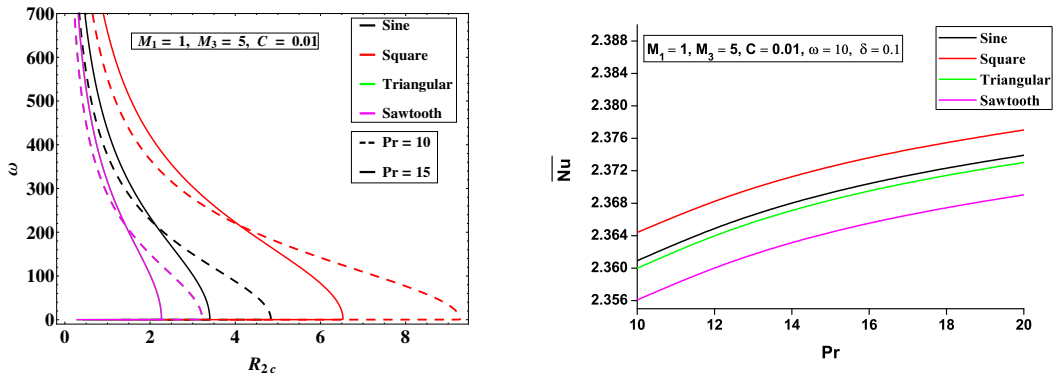
(A) (B)

FIGURE 4: Plot of (A) R_{2c} versus ω for different values of M_3 (B) $\overline{Nu}$ versus M_3 for sinusoidal and non-sinusoidal gravity modulation.



(A) (B)

FIGURE 5: Plot of (A) R_{2c} versus ω for different values of C (B) $\overline{Nu}$ versus C for sinusoidal and non-sinusoidal gravity modulation.



(A) (B)

FIGURE 6: Plot of (A) R_{2c} versus ω for different values of C (B) $\overline{Nu}$ versus C for sinusoidal and non-sinusoidal gravity modulation.

4 Individual effects of four types of rotation modulation on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress

In this section, we mainly focus on the effects of sine, square, triangular, and sawtooth wave type of rotation modulation (see section 1.2.2) on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress using linear and weakly non-linear analyses. Rotation modulation arises due to the uniform rotation of the fluid layer around the fixed z-axis in a time-periodic manner. The rotational modulation effects are introduced by the terms of Coriolis force: $2(\vec{q}\times\vec{\Omega}(t))$ and centrifugal force: $\frac{1}{2}\nabla|\vec{\Omega}(t)\times\vec{r}|^2$.

4.1 Mathematical formulation

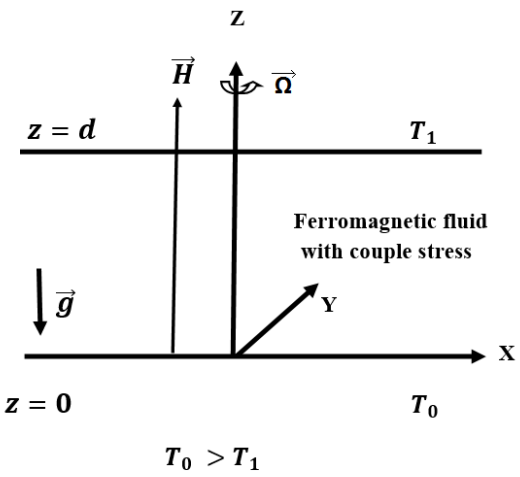


FIGURE 7: Physical configuration

The Boussinesq ferromagnetic fluid with couple stress rotating about the z-axis is considered between two parallel plates which are at a distance d from each other. Both the boundaries are taken to be free and perfect conductors of heat. Let T_o and $T_1(< T_o)$ be the temperatures at the lower and upper boundary respectively and $\Delta T = T_o - T_1$ be the temperature difference. A Cartesian coordinate system is considered with origin in the lower boundary with z-axis vertically upwards. The gravity acts vertically downwards (see Figure 7).

The governing equtaions of the problem is given by: (see [28, 41])

$$\nabla \cdot \vec{q} = 0, \tag{4.1}$$

$$\left.\begin{aligned} \rho_o \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} + 2\vec{\Omega}(t)\times\vec{q}\right] = -\nabla P + \rho\vec{g} + \mu_o(\vec{M}.\nabla)\vec{H} \\ +\mu\nabla^2\vec{q} - \mu'\nabla^4\vec{q} \end{aligned}\right\}, \tag{4.2}$$

$$\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T = \chi\nabla^2 T, \tag{4.3}$$

$$\rho = \rho_o[1 - \alpha_t(T - T_o)], \tag{4.4}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{0}, \tag{4.5}$$

$$\vec{B} = \mu_o(\vec{H} + \vec{M}), \tag{4.6}$$

$$\vec{M} = \frac{\vec{H}}{H}\, M(H,T), \tag{4.7}$$

$$M(H,T) = M_o + \chi_m(H - H_o) - \chi_T(T - T_o), \tag{4.8}$$

where $\vec{\Omega}(t) = \Omega_o[1 + \epsilon\, f(\gamma,t)]\hat{k}$, $\vec{\Omega}$ is the angular velocity, $\vec{\Omega}_o$ is the angular velocity in the absence of modulation, ϵ is the amplitude of rotation modulation, γ is the frequency of rotation modulation, $f(\gamma,t)$ is the time-dependent rotation modulation, $\vec{g}$ is the acceleration due to gravity, $P = p - \frac{\rho_o}{2}|\vec{\Omega}(t)\times\vec{r}|^2$ is the reduced pressure. Others quantities are as defined in the section 3.1.

The non-dimensional form of the governing equations are in the form:

$$\left.\begin{aligned} \frac{1}{Pr}\left[\frac{\partial}{\partial t}(\nabla^2\psi)\right] = \sqrt{Ta}\,[1 + \epsilon\, f(\omega,t)]\,\frac{\partial v}{\partial z} - R\frac{\partial T}{\partial x} + \nabla^4\psi - C\nabla^6\psi \\ +RM_1\frac{\partial^2\phi}{\partial x\partial z} - RM_1\frac{\partial T}{\partial x} + RM_1\, J\left(T, \frac{\partial\phi}{\partial z}\right) + \frac{1}{Pr}J(\psi,\nabla^2\psi) \end{aligned}\right\}, \tag{4.9}$$

$$\frac{1}{Pr}\left(\frac{\partial v}{\partial t}\right) = -\sqrt{Ta}\,[1 + \epsilon\, f(\omega,t)]\,\frac{\partial\psi}{\partial z} + (1 - C\nabla^2)\nabla^2 v + \frac{1}{Pr}J(\psi,v), \tag{4.10}$$

$$\frac{\partial T}{\partial t} = \nabla^2 T - \frac{\partial\psi}{\partial x} + J(\psi,T), \tag{4.11}$$

$$M_3\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial T}{\partial z} = 0, \tag{4.12}$$

where $Ta = \left(\frac{2\rho_o\Omega_o d^2}{\mu}\right)^2$ is the Taylor number.

The four different types of sinusoidal and non-sinusoidal wave forms considered to study the effect of rotational modulation is as defined in section 3.1 of problem 1.

Equations (4.9)–(4.12) are solved subjected to stress-free, isothermal, and magnetic boundary conditions based on the assumption of infinite magnetic susceptibility boundary conditions:

$$\psi = \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^4\psi}{\partial z^4} = \frac{\partial v}{\partial z} = \frac{\partial^3 v}{\partial z^3} = T \ = \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \tag{4.13}$$

4.2 Linear stability analysis

The equations (4.9)–(4.12) are solved by using the following Fourier series representation:

$$\psi(t, x, z) = A_1(t) \ \sin(\pi\alpha x) \ \sin(\pi z), \tag{4.14}$$
$$T(t, x, z) = A_2(t) \ \cos(\pi\alpha x) \ \sin(\pi z), \tag{4.15}$$
$$v(t, x, z) = A_3(t) \ \sin(\pi\alpha x) \ \cos(\pi z), \tag{4.16}$$
$$\phi(t, x, z) = \frac{A_4(t)}{\pi} \ \cos(\pi\alpha x) \ \cos(\pi z), \tag{4.17}$$

where A_1, A_2, A_3 and A_4 are the time dependent amplitudes.

The third order linearized Lorenz model for studying the effect of rotation modulation on the onset of convection is obtained by substituing equations (4.14)-(4.17) in the equations (4.9)-(4.12) and eliminating the amplitude $A_4(t)$ from the resulting equations.

$$\frac{dA_1}{dt} = -Pr\eta k^2 A_1 - \frac{R\pi\alpha Pr\mathcal{M}}{k^2} A_2 + \frac{\sqrt{Ta}\ \pi Pr}{k^2}[1 + \epsilon\ f(\omega, t)]A_3, \tag{4.18}$$
$$\frac{dA_2}{dt} = -\pi\alpha\ A_1 - k^2\ A_2, \tag{4.19}$$
$$\frac{dA_3}{dt} = -\sqrt{Ta}\ \pi Pr[1 + \epsilon\ f(\omega, t)]A_1 - Pr\eta k^2 A_3. \tag{4.20}$$

Following the analysis of multiscale method, we get:

$$R_o = \frac{k^6\eta}{\pi^2\alpha^2\mathcal{M}} + \frac{Ta}{\mathcal{M}\alpha^2\eta}, \tag{4.21}$$
$$R_1 = 0, \tag{4.22}$$
$$R_2 = -\frac{Tak^2Pr\epsilon_1^2 I_2}{\alpha^2\mathcal{M}} + \frac{Ta^2\epsilon_1^2\pi Pr}{\alpha^3\mathcal{M}(Y_1^2 + Y_2^2)}\left[\frac{\mathcal{I}}{\eta k^2} + PrI_2\right]\left[\frac{(k^2Y_1 - \omega Y_2)}{\eta} - \frac{Pr^2\eta k^2(k^2Y_1 - \omega Y_2) + \omega Pr(\omega Y_1 + k^2 Y_2)}{Pr^2k^4\eta^2 + \omega^2}\right], \tag{4.23}$$

where

$$Y_1 = \frac{\omega^2}{\pi\alpha} - \frac{k^4 Pr\eta}{\pi\alpha} + \frac{R_o\pi\alpha Pr\mathcal{M}}{k^2} - \frac{TaPr^2\pi(Pr\eta k^4 + \omega^2)}{k^2\alpha(Pr^2\eta^2k^4 + \omega^2)}, \quad I_1 = \int e^{Pr\eta k^2 t} f(\omega, t)dt,$$

$$Y_2 = \frac{\omega k^2}{\pi\alpha} + \frac{k^2 Pr\eta\omega}{\pi\alpha} - \frac{TaPr^2\pi(k^2\omega - \omega Pr\eta k^2)}{k^2\alpha(Pr^2\eta^2k^4 + \omega^2)}, \quad I_2 = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{f(\omega, t)I_1}{e^{Pr\eta k^2 t}}dt,$$

The following limiting cases can be recovered from equation (4.21).

- If $\mathcal{M} = \eta = 1$ and $Ta \neq 0$, equation (4.21) reduces to the expression obtained by Chandrashekar [68].
- If $\eta \neq 1, \mathcal{M} = 1$ and $Ta \neq 0$, equation (4.21) reduces to the expression obtained by Sharma and Sharma [69].
- If $\eta = 1, \mathcal{M} \neq 1$ and $Ta \neq 0$, equation (4.21) reduces to the expression obtained by Gupta and Gupta [22].
- If $Ta = 0, \mathcal{M} = 1$ and $\eta \neq 1$, equation (4.21) reduces to the expression obtained by Siddheshwar and Pranesh [41].

The critical Rayleigh number, R_c, is considerd to examine the effect of various parameters on the onset of convection and is given by:

$$R_c = R_{oc} + \epsilon^2 R_{2c}, \tag{4.24}$$

where R_{oc} is the critical Rayleigh number in the absence of rotation modulation and R_{2c} is the correction Rayleigh number.

4.3 Non-linear stability analysis

As linear stability analysis fails to predict the amount of heat transport, non-linear analysis is performed by representing the quantities using truncated Fourier Series to obtain the following fifth-order non-autonomous Lorenz model.

$$\frac{dA}{dt} = -ta\ Pr(1 + \epsilon f(\omega, t))E - Pr\ \eta A + Pr\mathcal{M}B + \Lambda BD, \tag{4.25}$$

$$\frac{dB}{dt} = -B + R'A - AD, \tag{4.26}$$

$$\frac{dD}{dt} = -bD + AB, \tag{4.27}$$

$$\frac{dE}{dt} = ta\ Pr(1 + \epsilon f(\omega, t))A - Pr\eta E - AF, \tag{4.28}$$

$$\frac{dF}{dt} = -dPrF - \Lambda_c PrF + AE, \tag{4.29}$$

where $b = \frac{4\pi^2}{k^2}$, $d = \alpha^2 b$, $\Lambda_c = \frac{16\pi^4\alpha^4 C}{k^2}$, $ta = \frac{\sqrt{Ta}\ \pi}{k^3}$, and $\Lambda = \frac{M_1 Pr}{(1 + M_3\alpha^2)R'}$.

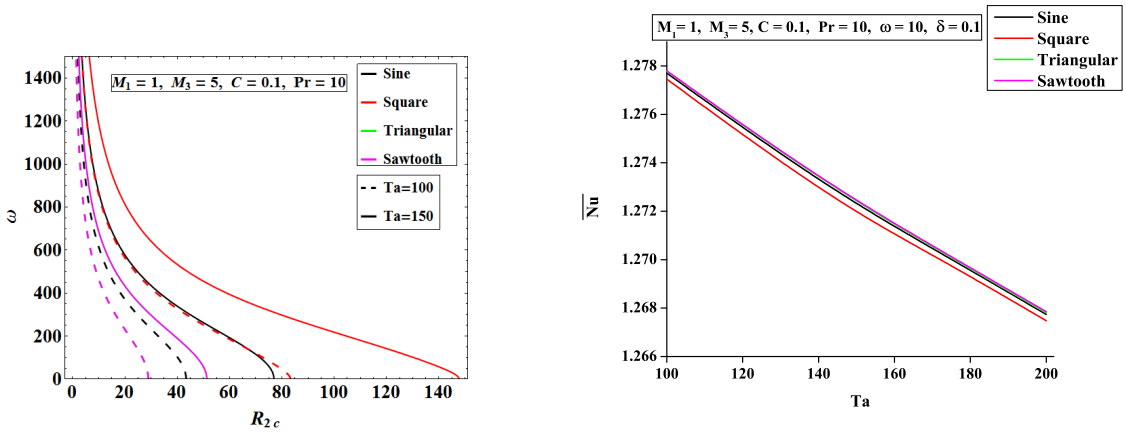
The classical Lorenz system [67] can be recovered by setting $M_1 = C = Ta = 0$ and $\epsilon = 0$ in equations (4.25)-(4.29).

Average Nusselt number is computed in order to study the effect of non-dimensional parameters on heat transfer and is given by:

$$\overline{Nu} = \left(\frac{\omega}{2\pi}\right) \int_0^{\frac{2\pi}{\omega}} Nu(t)\ dt, \tag{4.30}$$

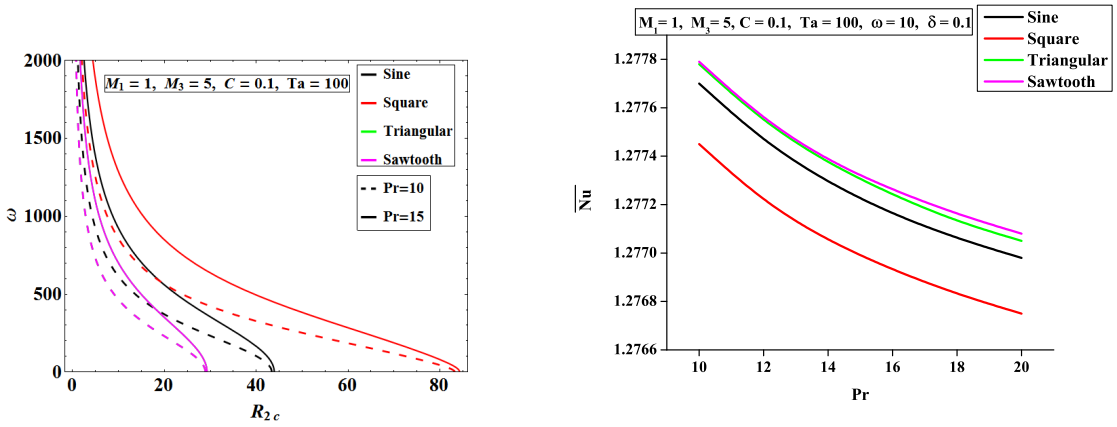
where

$$Nu(t) = 1 + \frac{2}{R'} D(t).$$



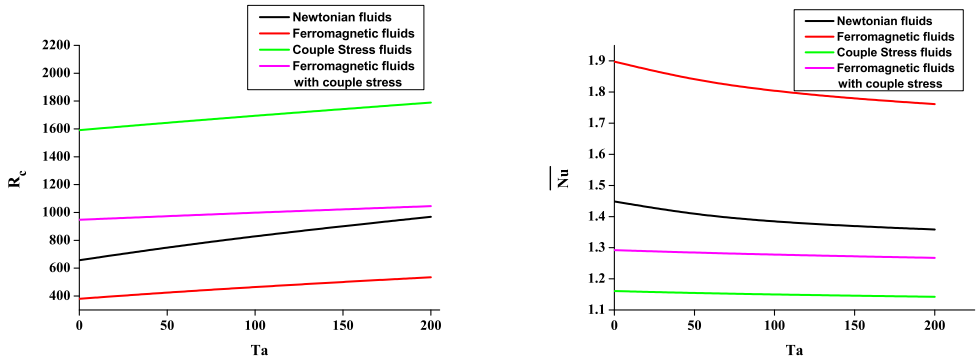
(A) (B)

FIGURE 8: Plot of (A) R_{2c} versus ω for different values of Ta (B) $\overline{Nu}$ versus Ta for sinusoidal and non-sinusoidal rotation modulation



(A) (B)

FIGURE 9: Plot of (A) R_{2c} versus ω for different values of Pr (B) $\overline{Nu}$ versus Pr for sinusoidal and non-sinusoidal rotation modulation



(A) (B)

FIGURE 10: Plot of (A) R_c versus Ta (B) $\overline{Nu}$ versus Ta for sinusoidal rotation modulation.

4.4 Results

The following conclusions are drawn from the study:

1. The effect of different parameters of the problem on R_c and $\overline{Nu}$ can be summarized as follows:
   (a) $R_c|Ta = 0 < R_c|Ta \neq 0, \quad \overline{Nu}|Ta = 0 > \overline{Nu}|Ta \neq 0$
   (b) $R_c|C = 0 < R_c|C \neq 0, \quad \overline{Nu}|C = 0 > \overline{Nu}|C \neq 0$
   (c) $R_c|M_1 = 0 > R_c|M_1 \neq 0, \quad \overline{Nu}|M_1 = 0 < \overline{Nu}|M_1 \neq 0$
   (d) $R_c|\delta = 0 < R_c|\delta \neq 0, \quad \overline{Nu}|\delta = 0 > \overline{Nu}|\delta \neq 0$
   (e) $R_c^{\text{Square}} > R_c^{\text{Sine}} > R_c^{\text{Triangular}} = R_c^{\text{Sawtooth}}$
   (f) $\overline{Nu}^{\text{Sawtooth}} > \overline{Nu}^{\text{Triangular}} > \overline{Nu}^{\text{Sine}} > \overline{Nu}^{\text{Square}}$
   (g) $R_c^{\text{Couple stress fluid}} > R_c^{\text{Ferromagnetic fluid with couple stress}} > R_c^{\text{Newtonian fluid}} > R_c^{\text{Ferromagnetic fluid}}$
   (h) $\overline{Nu}^{\text{Couple stress fluid}} < \overline{Nu}^{\text{Ferromagnetic fluid with couple stress}} < \overline{Nu}^{\text{Newtonian fluid}} < \overline{Nu}^{\text{Ferromagnetic fluid}}$
2. By proper tuning of frequency of rotation modulation it is possible to control the onset of Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress depending on the need of the application.
3. The effect of Ta, C and Pr is to stabilize the system and reduce the heat transport.
4. The effect of M_1 and M_3 is to destabilize the system and enhance the heat transport.
5. The effect of ω is to enhance the heat transport.
6. The effect of δ is to diminish the heat transport.
7. Maximum heat transport can be obtained by choosing the sawtooth waveform and least heat transport by choosing square waveform.
8. Square waveform is most stable compared to other waveforms.
9. Couple stress fluid most stabilizing and ferromagnetic fluids is most destabilizing in the case of sinusoidal rotation modulation.
10. Ferromagnetic fluid has the least diminishing effect on heat transport and couple stress fluid diminishes the most in the case of sinusoidal rotation modulation.

5 Effect of sinusoidal and non-sinusoidal temperature modulation on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress

The individual impact of trigonometric sine, square, triangular and sawtooth time-dependent boundary temperature (see section 1.2.3) on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress is analyzed by performing linear and weakly non-linear analyses in this section. Three cases of time-dependent boundary temperature (in phase, lower wall and out-of-phase) are considered to examine the influence of waveforms on the convection onset and heat transport. Temperature modulation in a ferromagnetic fluid with couple-stress is achieved by considering time-dependent lower plate and upper plate.

5.1 Mathematical formulation

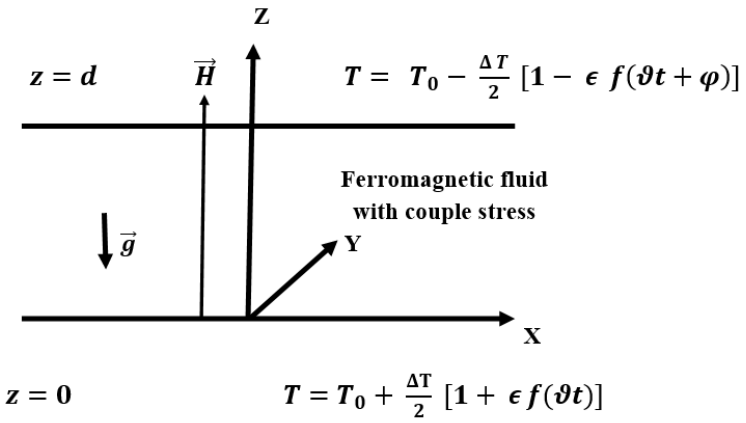


FIGURE 11: Physical Configuration

The physical configuration of the problem consists of an infinite horizontal extent of a ferromagnetic fluid with couple stress between two isothetmal, stress-free parallel plates at $z = 0$ and $z = d$. The Cartesian frame of reference is chosen in such a way that the z-axis is vertically upwards and origin lies on the lower plate. The set-up of the problem is as shown in Figure (11).

The equation of continuity, motion, energy and density, the Maxwell and magnetization equations is given by: (see [29, 41])

$$\nabla \cdot \vec{q} = 0, \tag{5.1}$$

$$\rho_o \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q}\right] = -\nabla p - \rho g \hat{k} + \mu_o (\vec{M}.\nabla)\vec{H} + \mu \nabla^2 \vec{q} - \mu' \nabla^4 \vec{q}, \tag{5.2}$$

$$\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T = \chi \nabla^2 T, \tag{5.3}$$

$$\rho = \rho_o [1 - \alpha_t (T - T_o)], \tag{5.4}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{0}, \tag{5.5}$$

$$\vec{B} = \mu_o (\vec{H} + \vec{M}), \tag{5.6}$$

$$\vec{M} = \frac{\vec{H}}{H} \left[M_o + \chi_m (H - H_o) - \chi_T (T - T_o)\right], \tag{5.7}$$

The temperature boundary conditions are given by:

$$T(0, t) = T_o + \frac{\Delta T}{2}[1 + \epsilon\, f(\vartheta t)], \tag{5.8}$$

$$T(d, t) = T_o - \frac{\Delta T}{2}[1 - \epsilon\, f(\vartheta t + \varphi)], \tag{5.9}$$

where, $f(\vartheta t)$ is the temperature modulation, ϵ is the amplitude of modulation, ϑ is frequency of modulation, φ is phase angle.

The three cases of temperature modulation considered in this study are:

case (i) In-phase ($\varphi = 0$),

case (ii) Lower-wall ($\varphi = -i\infty$),

case (iii) Out-of-phase ($\varphi = \pi$).

The dimensionless governing equations are given by:

$$\left.\begin{aligned}\frac{1}{Pr}\left[\frac{\partial}{\partial t}(\nabla^2\psi)\right] = \nabla^4\psi - C\nabla^6\psi - [-1+\epsilon\ \mathcal{G}]RM_1\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial T}{\partial x}\right] \\ -R\frac{\partial T}{\partial x} + RM_1\ J\left(T, \frac{\partial\phi}{\partial z}\right) + \frac{1}{Pr}J(\psi, \nabla^2\psi)\end{aligned}\right\}, \tag{5.10}$$

$$\frac{\partial T}{\partial t} = [-1+\epsilon\ \mathcal{G}]\frac{\partial\psi}{\partial x} + \nabla^2 T + J(\psi, T), \tag{5.11}$$

$$M_3\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial T}{\partial z} = 0, \tag{5.12}$$

where

$$\mathcal{G} = Im\ [E_1(m)e^{mz} + E_1(-m)e^{-mz}]f_j(\omega, t), \quad (j = 1, 2, 3) m = (1-i)\left(\frac{\omega}{2}\right)^{1/2}$$

$$\mathcal{G} = Re\ [E_1(m)e^{mz} + E_1(-m)e^{-mz}]f_4(\omega, t), \quad E_1(m) = \frac{m}{2}\left[\frac{e^{-i\varphi} - e^{-m}}{e^m - e^{-m}}\right].$$

The four types of sinusoidal and non-sinusoidal wave forms considered to study the effect of different cases of temperature modulation are as shown in section 3.1.

Equations (5.10)-(5.12) are solved subjected to the boundary condition:

$$\psi = \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^4\psi}{\partial z^4} = T\ = \frac{\partial\phi}{\partial z} = 0 \quad at \quad z = 0, 1. \tag{5.13}$$

5.2 Linear stability analysis

The expression for unmoudulated Rayleigh number and correction Rayleigh number is calculated by the Venezian approch [49].

$$R_o = \frac{k^6\eta}{\pi^2\alpha^2\mathcal{M}}, \tag{5.14}$$

$$R_1 = 0, \tag{5.15}$$

$$\left.\begin{aligned}R_2 = -R_o^2\pi^4\alpha^2\left[\frac{1 + M_3\alpha^2 + 2M_1M_3\alpha^2}{1 + M_3\alpha^2 + M_1M_3\alpha^2}\right]\mathcal{I} \\ \sum_{n=1}^{\infty}\frac{[M_3\alpha^2 + 2M_1M_3\alpha^2 + n^2]}{|\mathcal{L}(\omega, n)|^2}|\mathcal{B}_n(m)|^2\left[\frac{\mathcal{L}(\omega, n)| + \mathcal{L}^*(\omega, n)|}{2}\right]\end{aligned}\right\}, \tag{5.16}$$

where

$$\mathcal{L}(\omega, n) = Y_1 + iY_2,\ k_n^2 = n^2\pi^2 + \pi^2\alpha^2,\ \mathcal{B}_n(m) = \frac{-4nm^2\pi^2[(1 + (-1)^n e^{-i\varphi})]}{2[m^2 + (n+1)^2\pi^2][m^2 + (n-1)^2\pi^2]},$$

$$Y_1 = k_n^2(M_3\pi^2\alpha^2 + n^2\pi^2)\left(k_n^4 + Ck_n^6 - \frac{\omega^2}{Pr}\right) - R_o\pi^2\alpha^2\left[(1+M_1)(M_3\pi^2\alpha^2 + n^2\pi^2) - M_1n^2\pi^2\right],$$

$$Y_2 = -\omega k_n^4\left(1 + Ck_n^2 + \frac{1}{Pr}\right)(M_3\pi^2\alpha^2 + n^2\pi^2).$$

TABLE 2: Limiting cases in the absence of temperature modulation

Parameter	Critical Rayleigh number	Fluid	Author
$\mathcal{M} = \eta = 1$	$R_{oc} = \frac{k^6}{\pi^2\alpha^2}$	Newtonian fluid	Rayleigh [4]
$\eta = 1; \mathcal{M} \neq 1$	$R_{oc} = \frac{k^6}{\pi^2\alpha^2\mathcal{M}}$	Newtonian Ferromagnetic fluid	Finlayson [21]
$\mathcal{M} = 1; \eta \neq 1$	$R_{oc} = \frac{k^6\eta}{\pi^2\alpha^2}$	Couple stress fluid	Siddheshwar and Pranesh [41]

The critical Rayleigh number, R_c , is considerd to examine the effect of various parameters on the onset of convection and is given by:

$$R_c = R_{oc} + \epsilon^2 R_{2c}, \tag{5.17}$$

where R_{oc} is the critical Rayleigh number in the absence of temperature modulation and R_{2c} is the correction Rayleigh number.

5.3 Non-linear stability analysis

The tri-modal non-autonomous Lorenz model for analyzing the effect of all three cases of temperature modulation on the heat transport is obtained in the form:

$$\frac{dA}{dt} = -Pr\ \eta A + Pr\Gamma_{_M} B + \Lambda\ BD, \tag{5.18}$$

$$\frac{dB}{dt} = -B + R'A - \epsilon\mathcal{G}R'A - AD, \tag{5.19}$$

$$\frac{dD}{dt} = -bD + AB, \tag{5.20}$$

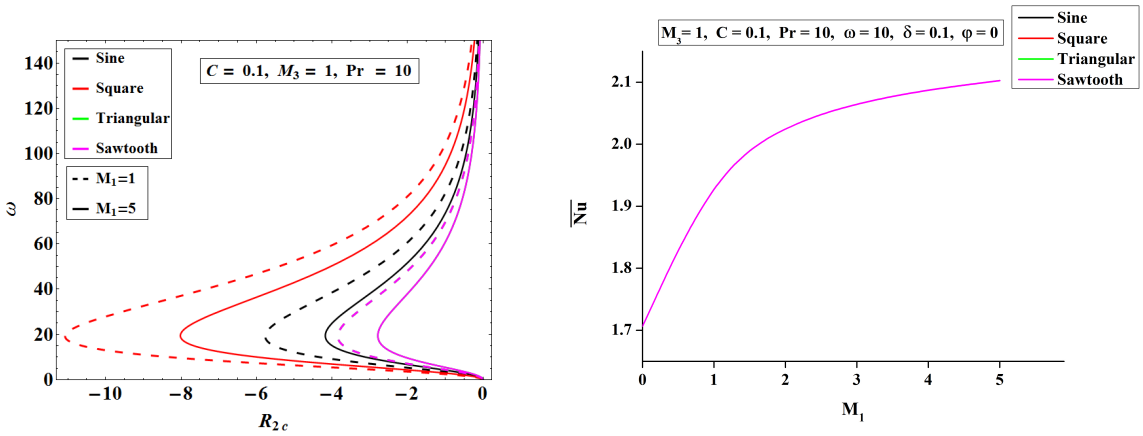
where $\Gamma_{_M} = \dfrac{1 + M_3\alpha^2 - (-1 + \epsilon\mathcal{G})M_1M_3\alpha^2}{1 + M_3\alpha^2}$, and $\Lambda = \dfrac{M_1Pr}{(1 + M_3\alpha^2)R'}$.

Average Nusselt number is computed in order to study the effect of non-dimensional parameters on heat transfer and is given by:

$$\overline{Nu} = \left(\frac{\omega}{2\pi}\right)\int_0^{\frac{2\pi}{\omega}} Nu(t)\ dt, \tag{5.21}$$

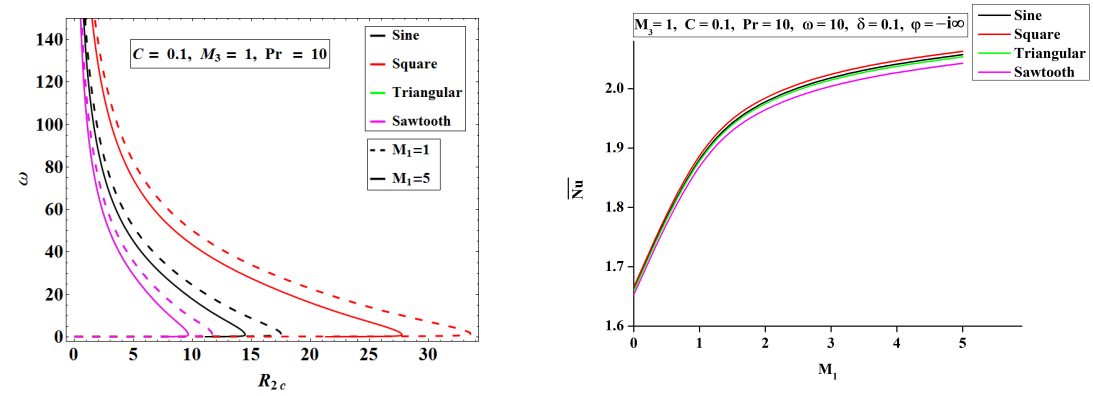
where

$$Nu(t) = 1 + \frac{2}{R'}D(t).$$



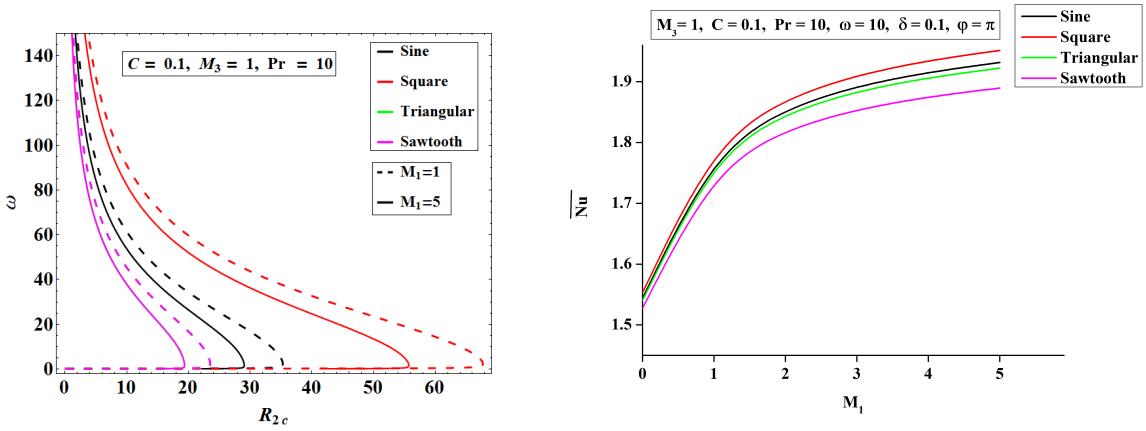
(A) (B)

FIGURE 12: Plot of (A) R_{2c} versus ω for different values of M_1 (B) $\overline{Nu}$ versus M_1 for sinusoidal and non-sinusoidal in-phase modulation.



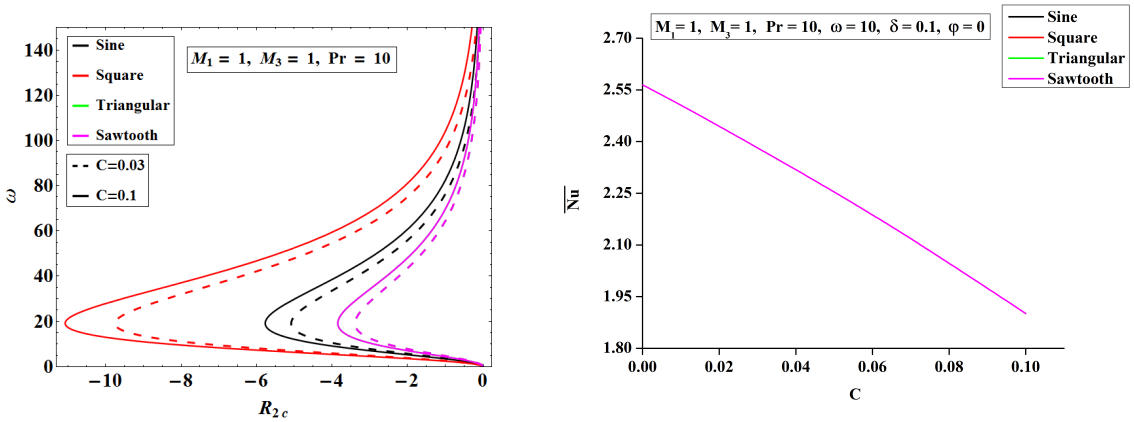
(A) (B)

FIGURE 13: Plot of (A) R_{2c} versus ω for different values of M_1 (B) $\overline{Nu}$ versus M_1 for sinusoidal and non-sinusoidal lowerwall modulation.



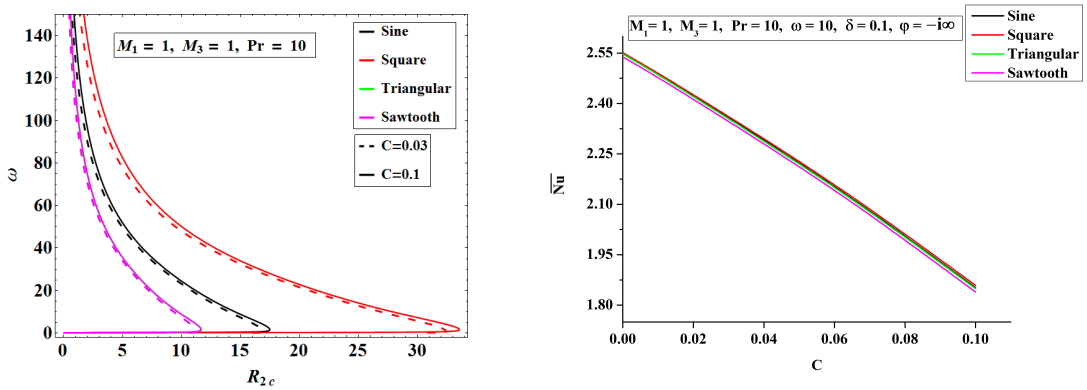
(A) (B)

FIGURE 14: Plot of (A) R_{2c} versus ω for different values of M_1 (B) $\overline{Nu}$ versus M_1 for sinusoidal and non-sinusoidal out-of-phase modulation.



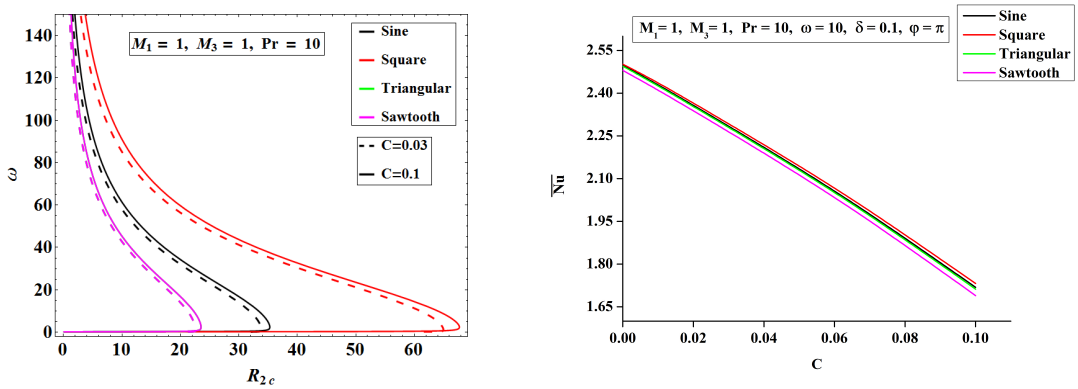
(A) (B)

FIGURE 15: Plot of (A) R_{2c} versus ω for different values of C (B) $\overline{Nu}$ versus C for sinusoidal and non-sinusoidal in-phase modulation.



(A) (B)

FIGURE 16: Plot of (A) R_{2c} versus ω for different values of C (B) $\overline{Nu}$ versus C for sinusoidal and non-sinusoidal lowerwall modulation.



(A) (B)

FIGURE 17: Plot of (A) R_{2c} versus ω for different values of C (B) $\overline{Nu}$ versus C for sinusoidal and non-sinusoidal out-of-phase modulation.

5.4 Results

The conclusions drawn from the study can be summarized as follows:

1. Temperature modulation can either stabilize or destabilize the system.
2. Inphase temperature modulation destabilizes the system whereas lower wall and out-of-phase temperature modulation stabilizes the system in comparision with the unmodulated system.
3. Out of phase temperature modulation is most stable than inphase and lowerwall temperature modulation, i.e., $R_c^{\text{in phase}} < R_c^{\text{lower wall}} < R_c^{\text{out of phase}}$.
4. Inphase temperature modulation facilitates more heat transport in comparision with lowerwall and out-of-phase temperature modulation, i.e, $\overline{Nu}^{\text{in phase}} > \overline{Nu}^{\text{lowerwall}} > \overline{Nu}^{\text{out of phase}}$.
5. The effect of C is to stabilize the system and reduce the heat transport.
6. The effect of M_1, M_3 and Pr is to destabilize the system and enhance the heat transport.
7. The effect of all four wavetypes of temperature modulation on R_{2c} and $\overline{Nu}$ can be summarized as follows:
   (a) $R_{2c}^{\text{Square}} < R_{2c}^{\text{Sine}} < R_{2c}^{\text{Triangular}} = R_{2c}^{\text{Sawtooth}}$
   (for inphase temperature modulation)
   (b) $R_{2c}^{\text{Square}} > R_{2c}^{\text{Sine}} > R_{2c}^{\text{Triangular}} = R_{2c}^{\text{Sawtooth}}$
   (for lowerwall and out-of-phase temperature modulation)
   (c) $\overline{Nu}^{\text{Square}} = \overline{Nu}^{\text{Sine}} = \overline{Nu}^{\text{Triangular}} = \overline{Nu}^{\text{Sawtooth}}$
   (for inphase temperature modulation)
   (d) $\overline{Nu}^{\text{Square}} > \overline{Nu}^{\text{Sine}} > \overline{Nu}^{\text{Triangular}} > \overline{Nu}^{\text{Sawtooth}}$
   (for lowerwall and out-of-phase temperature modulation)
8. Square wave enhnaces the heat transport and stabilizes the system in comparision with other waveforms.

6 Effect of sinusoidal and non-sinusoidal internal heat modulation on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress

In this section, we make a detailed study on Rayleigh-Bénard convection in a ferromagnetic fluid with couple-stress subjected to internal heat modulation (see section 1.2.4). The individual influence of sine, square, triangular and sawtooth waveform on the onset of convection and heat transport is analyzed by performing linear and weakly non-linear analyses.

6.1 Mathematical formulation

The physical configuration of the problem consists of an infinite horizontal extent of a ferromagnetic fluid with couple stress between two isothetmal, stress-free parallel plates at $z = 0$ and $z = d$. The Cartesian frame of reference is chosen in such a way that the z-axis is vertically upwards and origin lies on the lower plate. The set-up of the problem is as shown in Figure (18).

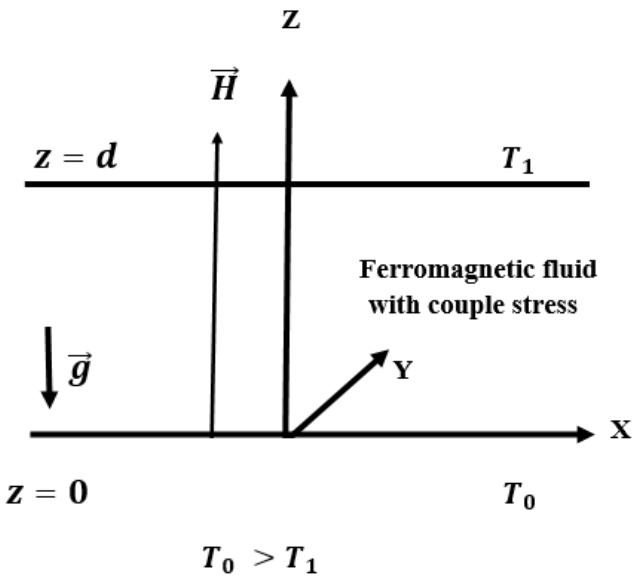


FIGURE 18: Physical Configuration

The conservation equations of mass, linear momentum and energy, the equation of state and Maxwell's equation in the presence of the applied magnetic field, constitutive equation and magnetic equation of state are (see [24, 41])

$$\nabla \cdot \vec{q} = 0, \tag{6.1}$$

$$\rho_o \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla p - \rho g \hat{k} + \mu_o(\vec{M}.\nabla)\vec{H} + \mu\nabla^2\vec{q} - \mu'\nabla^4\vec{q}, \tag{6.2}$$

$$\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T = \chi\nabla^2 T + Q\ (1 + \epsilon f(\gamma, t))(T - T_0), \tag{6.3}$$

$$\rho = \rho_o[1 - \alpha_t(T - T_o)], \tag{6.4}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{0}, \tag{6.5}$$

$$\vec{B} = \mu_o(\vec{H} + \vec{M}), \tag{6.6}$$

$$\vec{M} = \frac{\vec{H}}{H}\ \left[M_o + \chi_m(H - H_o) - \chi_T(T - T_o)\right], \tag{6.7}$$

where Q is the uniformly distributed internal heat source strength, T_o is the reference temperature and $f(\gamma, t)$ is the sinusoidal and non-sinusoidal wave forms of internal heat modulation.

The dimensionless governing equations are given by:

$$\left.\begin{aligned}\frac{1}{Pr}\left[\frac{\partial}{\partial t}(\nabla^2\psi)\right] = -R\frac{\partial T}{\partial x} + \nabla^4\psi - C\nabla^6\psi + RM_1 g(z)\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial T}{\partial x}\right] \\ +RM_1\ J\left(T, \frac{\partial\phi}{\partial z}\right) + \frac{1}{Pr}J(\psi, \nabla^2\psi)\end{aligned}\right\}, \tag{6.8}$$

$$\frac{\partial T}{\partial t} = -g(z)\frac{\partial\psi}{\partial x} + \nabla^2 T + R_i[1 + \epsilon f(\omega, t)]T + J(\psi, T), \tag{6.9}$$

$$M_3\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{\partial T}{\partial z}=0, \tag{6.10}$$

where $g(z)=\dfrac{\sqrt{R_i}}{\sin(\sqrt{R_i})}\cos(\sqrt{R_i}\ z)$ and $R_i=\dfrac{Qd^2}{\chi}$ is the internal Rayleigh number.

The four different types of internal heat modulation as given in section 3.1.

Equations (6.8) – (6.10) are solved subjected to the boundary conditions:

$$\psi=\frac{\partial^2\psi}{\partial z^2}=\frac{\partial^4\psi}{\partial z^4}=T\ =\frac{\partial\phi}{\partial z}=0 \qquad at \qquad z=0,1. \tag{6.11}$$

6.2 Linear stability analysis

The expression for unmoudulated Rayleigh number and correction Rayleigh number is calculated by the Venezian method [49].

$$R_o=\frac{k^4\eta[k^2-R_i][4\pi^2-R_i^2]^2}{\pi^2\alpha^2(4\pi^2\mathcal{M}-R_i)4\pi^2}, \tag{6.12}$$

$$R_1=0, \tag{6.13}$$

$$R_2=\frac{-R_i^2k^4\eta(1+M_3\alpha^2)\mathcal{I}}{\alpha^2G(1+M_3\alpha^2+M_1M_3\alpha^2G)}\sum_{n=1}^{\infty}\frac{[M_3\alpha^2+n^2]\left(k_n^4Y_1+Ck_n^6Y_1-\frac{\omega k_n^2Y_2}{Pr}\right)}{|\mathcal{L}(\omega,n)|^2}, \tag{6.14}$$

where $\mathcal{L}(\omega,n)=Y_1+iY_2, \quad k_n^2=n^2\pi^2+\pi^2\alpha^2, \quad G=\dfrac{-4\pi^2}{R_i-4\pi^2},$

$$\begin{aligned}Y_1&=(M_3\alpha^2\pi^2+n^2\pi^2)\left(\frac{-\omega^2k_n^2}{Pr}+k_n^6+Ck_n^8-R_ik_n^4-R_iCk_n^6\right)\\&\quad+R_o\pi^2\alpha^2G\left[(1+M_1G)(M_3\pi^2\alpha^2+n^2\pi^2)-M_1\pi^2n^2G\right],\\Y_2&=-\omega k_n^2(M_3\pi^2\alpha^2+n^2\pi^2)\left(k_n^2+Ck_n^4+\frac{k_n^2}{Pr}-\frac{R_i}{Pr}\right),\end{aligned}$$

The following limiting cases can be recovered from equation (6.12).

- If $\mathcal{M}=\eta=1$ and $R_i\neq0$, equation (6.12) reduces to the expression obtained by Sparrow et al. [70].
- If $\eta=1,\mathcal{M}\neq1$ and $R_i\neq0$, equation (6.12) reduces to the expression obtained by Rudraiah and Sekhar [24].

The critical Rayleigh number, R_c , is considerd to examine the effect of various parameters on the onset of convection and is given by:

$$R_c=R_{oc}+\epsilon^2R_{2c}, \tag{6.15}$$

where R_{oc} is the critical Rayleigh number in the absence of internal heat modulation and R_{2c} is the correction Rayleigh number.

6.3 Non-linear stability analysis

The generalized Lorenz model for analyzing the effect of sinusoidal and non-sinusoidal internal heat modulation on the heat transport is given by:

$$\frac{dA}{dt} = -\eta PrA + Pr\zeta B + \Lambda BD, \tag{6.16}$$

$$\frac{dB}{dt} = -B + \frac{R_i}{k^2}[1 + \epsilon f(\omega, t)]B + GR'A - AD, \tag{6.17}$$

$$\frac{dD}{dt} = -bD + \frac{R_i}{k^2}[1 + \epsilon f(\omega, t)]D + AB, \tag{6.18}$$

where $\zeta = 1 - \dfrac{4M_1M_3\pi^2\alpha^2}{[R_i - 4\pi^2][1 + M_3\alpha^2]}$ and $G = \dfrac{-4\pi^2}{R_i - 4\pi^2}$.

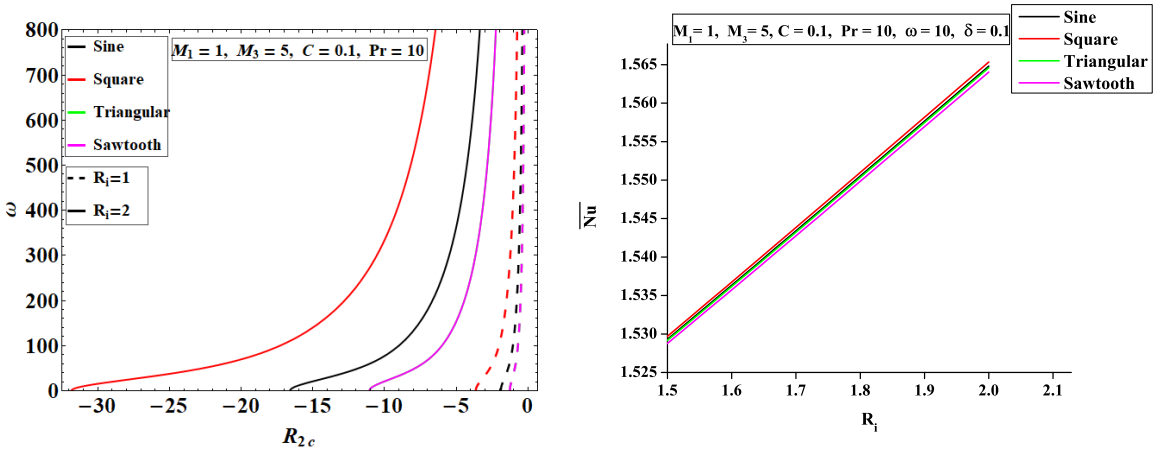
Average Nusselt number is considered in order to study the effect of various parameters on heat transfer and is given by:

$$\overline{Nu} = \left(\frac{\omega}{2\pi}\right) \int_0^{\frac{2\pi}{\omega}} \left[1 + \frac{2}{R'}D(t)\right] \, dt. \tag{6.19}$$

6.4 Results

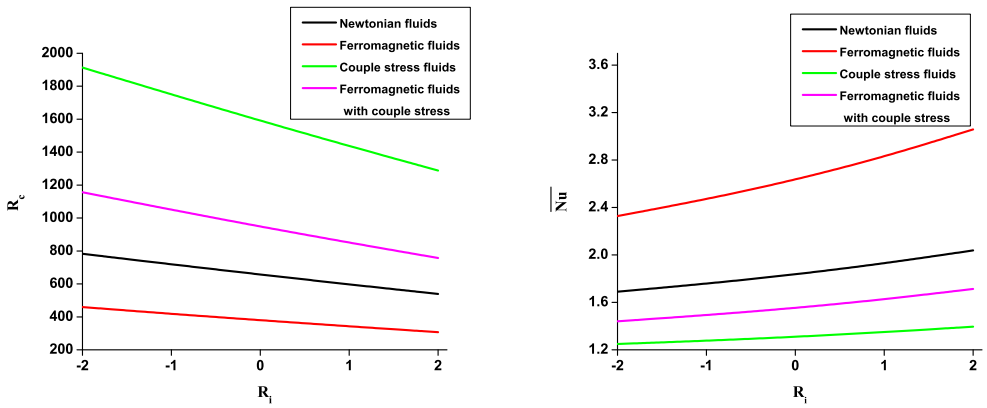
The conclusions drawn from the study can be summarized as follows:

1. Internal heat modulation in a ferromagnetic liquid with couple stress destabilizes the system and enhances heat transport in comparision with the unmodulated system.
2. It is possible to control the onset of convection and heat transport by choosing any of the four wave forms of internal heat modulation.
3. Increase in the value of the couple stress parameter C stabilizes the system and decreases the heat transport.
4. Increase in M_1, M_3 and R_i destabilizes the system and enhances the heat transport.
5. Sawtooth wave form is more stable compared to sine, triangular and square wave forms, i.e., $R_c^{\text{Square}} < R_c^{\text{Sine}} < R_c^{\text{Triangular}} = R_c^{\text{Sawtooth}}$.
6. Square wave form enhances the heat transport compared to sine, triangular and sawtooth wave forms for positive values of R_i.
7. Sawtooth wave form enhances the heat transport compared to sine, triangular and square wave forms for negative values of R_i.
8. Ferromagnetic fluid has the least diminishing effect on heat transport in comparision with Newtonian fluids, couple stress fluids and ferromagnetic fluids with couple stress .



(A) (B)

FIGURE 19: Plot of (A) R_c versus ω for different values of R_i (B) $\overline{Nu}$ versus R_i for sinusoidal and non-sinusoidal internal heat modulation.



(A) (B)

FIGURE 20: Plot of (A) R_c versus R_i (B) $\overline{Nu}$ versus R_i for Newtonian fluids, Couple stress fluids, Ferromagnetic fluids and Ferromagnetic fluid with couple stress.

7 Analogy for Rayleigh-Bénard convection between ferromagnetic fluid and dielectric fluid with couple stress

This section discusses an analogy for Rayleigh-Bénard convection between ferromagnetic fluid and dielectric fluid with couple stress. The non-dimensional governing equations for ferromagnetic fluids with couple stress in the absence of modulations ($\epsilon = 0$) are as shown in equations (3.8) - (3.10).

Siddheshwar [37] showed that there exists one-to-one correspondence between Rayleigh-Bénard convection in a ferromagnetic fluids and dielectric fluids. The non-dimensional govering equations for Rayleigh-Bénard convection in a dielectric fluid with couple stress is obatined by replacing M_1 by L and M_3 by 1 in the non-dimensional govering

equations of ferromagnetic fluid with couple stress.

$$\left.\begin{aligned}\frac{1}{Pr}\left[\frac{\partial}{\partial t}(\nabla^2\psi)\right] = -R\frac{\partial T}{\partial x} + RL\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial T}{\partial x}\right] + \nabla^4\psi - C\nabla^6\psi \\ +RL\ J\left(T, \frac{\partial\phi}{\partial z}\right) + \frac{1}{Pr}J(\psi, \nabla^2\psi)\end{aligned}\right\}, \tag{7.1}$$

$$\frac{\partial T}{\partial t} = \nabla^2 T - \frac{\partial\psi}{\partial x} + J(\psi, T), \tag{7.2}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial T}{\partial z} = 0, \tag{7.3}$$

where ϕ is the electric potential and L is the electric Rayleigh number.

The Rayleigh number and Nusselt number for dielectric fluid with couple stress can be easily obtained by replacing M_1 by L and M_3 by 1 respectively in the corresponding equations obtained in case of ferromagnetic fluid with couple stress. In view of the above analogy, there is no need to study the Rayleigh-Bénard convection in a dielectric fluid with couple stress in isolation.

8 Conclusion

The following observations are made from the study:

1. The ferromagnetic fluid with couple stress is more stable than Newtonian ferromagnetic fluid.
2. The buoyancy magnetization parameter and non-buoyancy magnetization parameter advances the onset of convection and destabilizes the system.
3. Heat transport increases with increase in M_1 and M_3.
4. The couple stress parameter stabilizes the system and diminishes the heat transport.
5. Square wave form is more stable compared to sine, triangular and sawtooth wave forms.
6. Gravity and rotation modulation in a ferromagnetic fluid with couple stress stabilizes the system and diminishes the heat transport.
7. Temperature modulation can either stabilize or destabilize the system.
8. Internal heat modulation in a ferromagnetic fluid with couple stress destabilizes the system and enhances the heat transport.

9 Scope for future research

In this research work, we have focused more on the regulation of modulated Rayleigh-Bénard convection through means of different waveforms in a ferromagnetic fluid with couple stress subjected to stress-free, isothermal boundary conditions. The effect of concentration, electric field and magnetic field modulation on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress at best will form our future work. The impact of modulated Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress subjected to realistic boundary conditions can be scope for future work.

Publications

1. J. Meghana, S. Pranesh, and P. G. Siddheshwar, "Individual effects of sinusoidal and non-sinusoidal gravity modulation on Rayleigh-Bénard convection in a ferromagnetic liquid and in a nanoliquid with couple stress," *Eur. Phys. J. Spec. Top.*, vol. 230, no. 5, pp. 1415–1425, 2021.
2. J. Meghana and S. Pranesh, "Individual effects of four types of rotation modulation on Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress," *Heat Transf.*, no. May, pp. 1–21, 2021, doi: 10.1002/htj.22204.

Presentations

1. Presented a paper entitled "Effect of Sinusoidal and Non-Sinusoidal Gravity Modulation on Rayleigh-Bénard Ferroconvection in Couple Stress Fluid" in the 64^{th} Congress of Indian Society of Theoretical and Applied Mechanics (ISTAM) during 9-12 December 2019 at IIT, Bhubaneswar, Odisha.

2. Presented a paper entitled "Effect of different wave types of time periodic Coriolis force on the onset of Rayleigh-Bénard convection in a ferromagnetic fluid with couple stress" in the International Conference on Emerging Trends in Fluid Mechanics and its Applications (ICETFMA-2020) during 27-28 February 2020 at CHRIST (Deemed to be University), Bangalore.

3. Presented a paper entitled "Effect of Synchronous and Asynchronous time-periodic Temperature on the onset of Rayleigh-Bénard Convection in a Ferromagnetic fluid with Couple stress"in 5^{th} International Conference on Applications of Fluid Dynamics (An Online Conference) (ICAFD-2020) during December 13-15, 2020 at VIT-AP University, India.

4. Presented a paper entitled "Linear and Non-Linear Analysis of Internal Heat Modulation on Rayleigh-Bénard Convection in Ferromagnetic Liquids with Couple Stress"in 86th Annual Conference of the Indian Mathematical Society (An International Meet) (IMS2020) during December 17-20, 2020 at VIT, Vellore.

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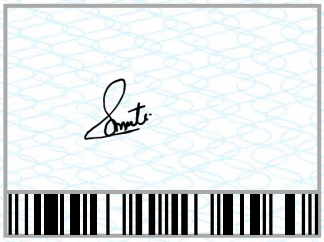

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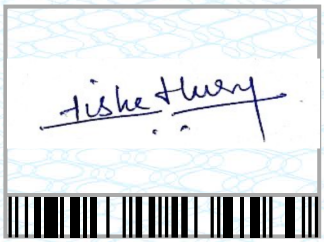

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